

The Primal versus the Dual Ising Model

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Abstract—We represent the Ising model of statistical physics by Forney factor graphs in the primal and in the dual domains. By analogy with Kirchhoff's voltage and current laws, we show that in the primal Forney factor graph, the dependency among the variables is along the cycles, whereas in the dual Forney factor graph, the dependency is on the cutsets. In the primal (resp. dual) domain, dependent variables can be computed via their fundamental cycles (resp. fundamental cutsets). In each domain, we propose an importance sampling algorithm to estimate the partition function. In the primal domain, the proposal distribution is defined on a spanning tree, and computations are done on the cospanning tree. In contrast, in the dual domain, computations are done on a spanning tree of the model, and the proposal distribution is defined on the cospanning tree.

I. INTRODUCTION

In this paper, we relate some properties of the Forney factor graph (FFG) [1] representation of the Ising model in the primal and in the dual domains to pertinent results in algebraic graph theory. The focus is on ferromagnetic Ising models with arbitrary topology and with pairwise (nearest-neighbor) interactions between the variables/particles.

Here, the FFG is a finite connected graph $G = (V, E)$, where V is the set of vertices and E the set of edges. In our analysis, we use partitionings of G into $G = T \cup \bar{T}$, where T is a spanning tree and \bar{T} is the corresponding cospanning tree. The edges of T are called the *branches* and the edges of \bar{T} are called the *chords* of G with respect to T , or simply the chords of T . Since G is connected, $|E| \geq |V| - 1$ and

$$|T| = |V| - 1 \quad (1)$$

$$|\bar{T}| = |E| - |V| + 1 \quad (2)$$

where $|\cdot|$ denotes the cardinality of a set.

We prove that the sum (modulo 2) of variables along any cycle in the primal FFG is zero. We then define a proposal (auxiliary) distribution on a spanning tree T to present an importance sampling algorithm for estimating the partition function. The algorithm can efficiently compute an estimate of the partition function when the coupling parameters of the Ising model associated with the chords are weak.

In the dual FFG, we prove that the sum (modulo 2) of variables on any cutset is zero. The proposal distribution is defined on \bar{T} . In this case, the partition function can be efficiently estimated when the coupling parameters associated with the branches are strong.

For more details on the cycle space, the cutset space, and their duality in the context of algebraic graph theory, see [2, Chapter 2], [3, Chapter 14].

Finally, we would like to point out that David Forney has recently developed some of the results of this paper (and their generalizations to the q -state Potts model) in the context of algebraic topology [4]. We encourage readers who are interested in the results of this paper to have a look at [4].

The paper is organized as follows. In Section II, we review the Ising model and its graphical model representation in terms of FFGs. In Section III, we describe our importance sampling algorithm in the primal domain. The dual FFG of the Ising model is discussed in Section IV. An analogous importance sampling in the dual FFG is presented in Section V.

II. THE ISING MODEL IN THE PRIMAL DOMAIN

Let us consider N interacting discrete random variables $\mathbf{X} = (X_1, X_2, \dots, X_N)$, where a realization of \mathbf{X} is denoted by \mathbf{x} . Suppose each random variable takes on values in a finite alphabet \mathcal{X} , which in this context is equal to the binary field \mathbb{F}_2 . A real coupling parameter $J_{k,\ell}$ is associated with each interacting pair (X_k, X_ℓ) .

Let $f: \mathcal{X}^N \rightarrow \mathbb{R}_{\geq 0}$ be a non-negative real function which factors into a product of local factors $v_{k,\ell}(\cdot)$ as

$$f(\mathbf{x}) = \prod_{(k,\ell) \in E} v_{k,\ell}(x_k, x_\ell) \quad (3)$$

where E contains all the unordered pairs (k, ℓ) with non-zero interactions, and $v_{k,\ell}: \mathcal{X}^2 \rightarrow \mathbb{R}_{\geq 0}$ is given by

$$v_{k,\ell}(x_k, x_\ell) = \begin{cases} e^{J_{k,\ell}}, & \text{if } x_k = x_\ell \\ e^{-J_{k,\ell}}, & \text{if } x_k \neq x_\ell \end{cases} \quad (4)$$

The model is called ferromagnetic (resp. antiferromagnetic) if $J_{k,\ell} > 0$ (resp. $J_{k,\ell} < 0$) for each $(k, \ell) \in E$.

From (3), we define the following probability mass function (also known as the Boltzmann distribution [5]) over \mathcal{X}^N

$$p_B(\mathbf{x}) \triangleq \frac{f(\mathbf{x})}{Z} \quad (5)$$

Here, the normalization constant Z is the *partition function* given by

$$Z = \sum_{\mathbf{x}} f(\mathbf{x}) \quad (6)$$

where the summation runs over all possible configurations.

The factorization in (3) can be represented by a FFG $G = (V, E)$, in which vertices represent the factors and edges represent the variables. The edge that represents some variable x is connected to the vertex representing the factor $v(\cdot)$ if and only if x is an argument of $v(\cdot)$. If a variable (an edge) appears

in more than two factors, such a variable is replicated using an equality indicator factor [1].

The primal FFGs of the Ising model on a chain (1D graph with periodic boundaries), a 2D lattice, and a fully-connected graph are shown in Fig. 1, where the unlabeled boxes represent (4) and boxes labeled “=” are equality indicator factors.

For example, the equality indicator factor $\Phi_{=}(·)$ involving variables x_2, x'_2 , and x''_2 is given by (see Fig. 1-left)

$$\Phi_{=}(x_2, x'_2, x''_2) = \delta(x_2 - x'_2) \cdot \delta(x_2 - x''_2) \quad (7)$$

where $\delta(·)$ is the Kronecker delta function.

We note that each factor (4) is only a function of $x_k + x_\ell$. (Recall that arithmetic manipulations are done modulo 2.) We can thus represent $v_{k,\ell}(·)$ using only one variable y_m , as

$$v_m(y_m) = \begin{cases} e^{J_m}, & \text{if } y_m = 0 \\ e^{-J_m}, & \text{if } y_m = 1 \end{cases} \quad (8)$$

Let \mathbf{Y} denote $(Y_1, Y_2, \dots, Y_{|E|})$, where $|\mathbf{Y}| = |E|$.

Following the above observation, we will construct the “modified” primal FFG of the Ising model as shown in Fig. 2, where the unlabeled boxes represent (8) and boxes labeled “+” are zero-sum indicator factors, which impose the constraint that all their incident variables sum to zero.

For example, the zero-sum indicator factor $\Phi_{+}(·)$ involving x_1, x_2 , and y_1 in Fig. 2-left is given by

$$\Phi_{+}(y_1, x_1, x_2) = \delta(y_1 + x_1 + x_2) \quad (9)$$

In the sequel, we drop the adjective “modified” and refer to the FFGs in Fig. 2 as the primal FFGs of the Ising model, when it causes no confusion.

By analogy with Kirchhoff’s voltage law, we prove:

Lemma 1. Consider a cycle in the primal FFG of an Ising model. If the variables attached to the zero-sum indicator factors along the cycle are Y_1, Y_2, \dots , then

$$\sum_{m \in \text{Cycle}} Y_m = 0 \quad (10)$$

Proof. We write each Y_m as the addition of its corresponding edges (X_k, X_ℓ) attached to the zero-sum indicator factor along the cycle (see (9)). Each variable, say X_k , will then appear twice in the summation. We conclude $\sum_{m \in \text{Cycle}} Y_m = 0$. ■

An example of a cycle is shown by thick edges in Fig. 3, where the variables Y_1, Y_2, \dots attached to the zero-sum indicator factors along the cycle are marked blue.

Let us partition G into $G = T \cup \bar{T}$, where T is a spanning tree in the primal FFG. Thus \mathbf{Y} will also be partitioned into $\mathbf{Y}_T \cup \mathbf{Y}_{\bar{T}}$. Examples of such partitionings are shown in Fig. 2, where spanning trees are marked by thick black edges, edges attached to the unlabeled boxes and to the zero-sum indicator factors on the branches represent \mathbf{Y}_T , and edges attached to the unlabeled boxes and to the zero-sum indicator factors on the chords represent $\mathbf{Y}_{\bar{T}}$.

For a given configuration \mathbf{y}_T , adding a chord $c \in \bar{T}$ to T will create a unique cycle called the fundamental cycle associated with c , which contains exactly one chord that does

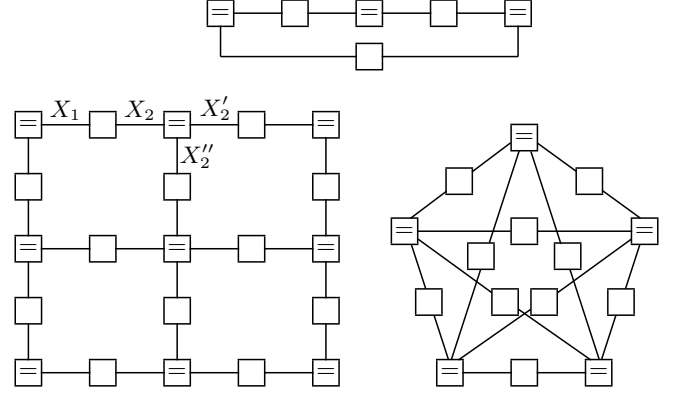


Fig. 1. The Primal FFG of the Ising model on a (top) chain (left) 2D lattice (right) fully-connected graph. The unlabeled boxes represent (4) and boxes containing “=” symbols are given by (7).

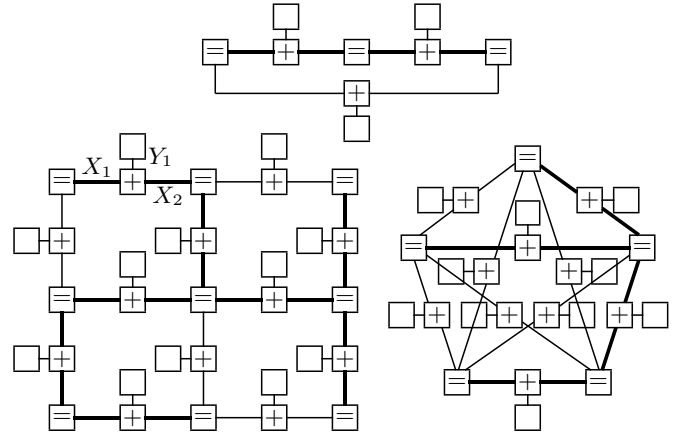


Fig. 2. Modified primal FFGs of the Ising models in Fig. 1. The unlabeled boxes represent (8), boxes containing “+” symbols are as in (9), and boxes containing “=” symbols are given by (7). In each FFG, the branches of a spanning tree are marked by thick black edges.

not appear in any other fundamental cycle¹. Furthermore, according to Lemma 1, for each $c \in \bar{T}$ we can compute y_c as a linear combination of \mathbf{y}_T . As a result, the size of the state space in the modified primal FFG is $|\mathcal{X}|^{|\bar{T}|} = |\mathcal{X}|^{N-1}$.

Remark 1. In the primal FFG, we can freely choose a configuration \mathbf{y}_T , and therefrom deterministically compute each component of $\mathbf{y}_{\bar{T}}$ via its fundamental cycle.

Accordingly, let

$$\Upsilon_T(\mathbf{y}_T) = \prod_{m \in T} v_m(y_m) \quad (11)$$

$$\Upsilon_{\bar{T}}(\mathbf{y}_{\bar{T}}) = \prod_{m \in \bar{T}} v_m(y_m) \quad (12)$$

and

$$\Upsilon(\mathbf{y}) = \Upsilon_T(\mathbf{y}_T) \cdot \Upsilon_{\bar{T}}(\mathbf{y}_{\bar{T}}) \quad (13)$$

¹Indeed, the set of all fundamental cycles generates a vector space over \mathbb{F}_2 with dimensionality $|\bar{T}|$; see [2, Chapter 2], [3, Chapter 14].

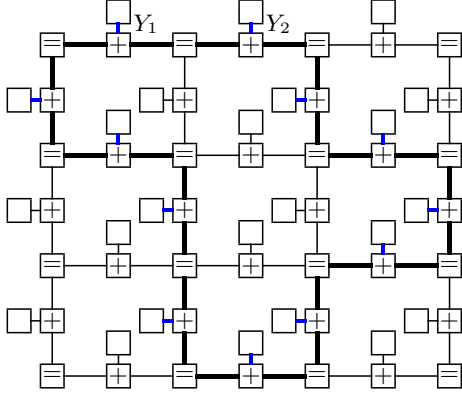


Fig. 3. The thick edges show a cycle in the primal FFG of the 2D Ising model, where variables attached to the zero-sum indicator factors along the cycle are marked blue.

The global probability mass function in the modified primal FFG can then be defined as

$$p_M(\mathbf{y}) \triangleq \frac{\Upsilon(\mathbf{y})}{Z_M} \quad (14)$$

where the partition function Z_M is given by

$$Z_M = \sum_{\mathbf{y}} \Upsilon(\mathbf{y}) \quad (15)$$

Lemma 2. The partition functions Z and Z_M are related by

$$Z = 2Z_M \quad (16)$$

Proof. Let $\neg \mathbf{x}$ be the component-wise addition of \mathbf{x} and the all-ones vector, i.e., in $\neg \mathbf{x}$, components of \mathbf{x} that are 0 become 1, and those that are 1 become 0. There are $|\mathcal{X}|^N$ configurations \mathbf{x} that contribute to Z in (6). Let us partition \mathcal{X}^N into \mathcal{X}_1 and \mathcal{X}_2 , where for each $\mathbf{x} \in \mathcal{X}_1$, we have $\neg \mathbf{x} \in \mathcal{X}_2$, and vice-versa. Note that $|\mathcal{X}_1| = |\mathcal{X}_2| = |\mathcal{X}|^{N-1}$.

There are $|\mathcal{X}|^{N-1}$ configurations \mathbf{y} in the modified primal FFG with non-zero contributions to Z_M in (15). From one such configuration \mathbf{y} , we can compute exactly two corresponding configurations \mathbf{x} and $\neg \mathbf{x}$ in the primal FFG (e.g., by setting $x_1 = 0$ and $x_1 = 1$ to solve a system of equations for \mathbf{x} and for $\neg \mathbf{x}$). However, due to symmetry in the factors in (4), the contribution of \mathbf{x} and of $\neg \mathbf{x}$ to Z is exactly $\Gamma(\mathbf{y})$. ■

To better show the contrast between the two domains, we factor out $\exp(J_m)$ from (8), and work with local factors as

$$v'_m(y_m) = e^{-2J_m \cdot y_m} \quad (17)$$

Similarly, we define $\Upsilon'_T(\cdot)$, $\Upsilon'_{\bar{T}}(\cdot)$, $p'_M(\cdot)$, and Z'_M according to the new local factors in (17), e.g.,

$$\Upsilon'_T(\mathbf{y}_T) = \prod_{m \in T} v'_m(y_m) \quad (18)$$

$$\Upsilon'_{\bar{T}}(\mathbf{y}_{\bar{T}}) = \prod_{m \in \bar{T}} v'_m(y_m) \quad (19)$$

From (16) and (17), the partition function in (6) can be expressed as

$$Z = 2A \cdot Z'_M \quad (20)$$

with $A = \exp(\sum_{m \in E} J_m)$.

In the next section, we propose an importance sampling algorithm in the primal FFG of the Ising model to compute an estimate of Z'_M , and hence of Z itself.

III. IMPORTANCE SAMPLING IN THE PRIMAL FFG

The importance sampling algorithm works as follows. We first draw *independent* samples $\mathbf{y}_T^{(1)}, \mathbf{y}_T^{(2)}, \dots$ according to a proposal probability mass function $q_T(\mathbf{y}_T)$ defined on T , and therefrom compute $\mathbf{y}_{\bar{T}}^{(1)}, \mathbf{y}_{\bar{T}}^{(2)}, \dots$. The created samples are then used to compute a Monte Carlo estimate of Z'_M .

The proposal probability mass function is defined as

$$q_T(\mathbf{y}_T) \triangleq \frac{\Upsilon'_T(\mathbf{y}_T)}{Z_{q_T}} \quad (21)$$

where Z_{q_T} is available in closed form as

$$Z_{q_T} = \sum_{\mathbf{y}_T} \Upsilon'_T(\mathbf{y}_T) = \prod_{m \in T} (1 + e^{-2J_m}) \quad (22)$$

Drawing independent samples according to $q_T(\mathbf{y}_T)$ is straightforward because $\Upsilon'_T(\cdot)$ decomposes into a product (cf. (18)). Therefore, we can draw each component $y_m^{(\ell)}$ of $\mathbf{y}_T^{(\ell)}$ independently according to $P_m(\cdot) \propto v'_m(\cdot)$. Indeed

$$P_m(y_m^{(\ell)} = \zeta) = \begin{cases} \frac{1}{1 + e^{-2J_m}} & \text{if } \zeta = 0 \\ \frac{e^{-2J_m}}{1 + e^{-2J_m}} & \text{if } \zeta = 1 \end{cases} \quad (23)$$

After drawing $\mathbf{y}_T^{(\ell)}$, we compute $\mathbf{y}_{\bar{T}}^{(\ell)}$, and then use L created samples in the following importance sampling estimator

$$\hat{Z}'_M = \frac{Z_{q_T}}{L} \sum_{\ell=1}^L \Upsilon'_{\bar{T}}(\mathbf{y}_{\bar{T}}^{(\ell)}) \quad (24)$$

which is an unbiased estimator of Z'_M , i.e., $\mathbb{E}_{q_T}[\hat{Z}'_M] = Z'_M$.

Suppose the coupling parameters on a cospanning tree of the model are constant $J_{\bar{T}}$. In the limit $J_{\bar{T}} \rightarrow 0$, $\Upsilon'_{\bar{T}}(\cdot)$ becomes a constant factor (cf. (17), (19)), therefore we expect the importance sampling estimator to perform well in the primal FFG when couplings on a cospanning tree are small.

The variance of (24) is given by [6]

$$\frac{L}{Z'^2_M} \text{Var}[\hat{Z}'_M] = \chi^2(p'_M, q_T) \quad (25)$$

Here, $\chi^2(\cdot, \cdot)$ denotes the chi-square distance, which is non-negative, with equality to zero if and only if its two arguments are equal [7, Chapter 4].

Note that $\lim_{J_{\bar{T}} \rightarrow 0} p'_M(\cdot) = q_T(\cdot)$. Hence

$$\lim_{J_{\bar{T}} \rightarrow 0} \chi^2(p'_M, q) = 0 \quad (26)$$

This result suggests that a plausible strategy in partitioning G is to include edges with weaker couplings in \bar{T} , which can be accomplished via the maximum spanning tree algorithm.

IV. THE ISING MODEL IN THE DUAL DOMAIN

We briefly discuss an analogous importance sampling algorithm to estimate the partition function in the dual domain.

The dual FFG has the same topology as the primal FFG, but factors are replaced by the discrete Fourier transform (DFT) of their corresponding factors in the primal FFG, and variables are replaced by their corresponding dual variables. We use the tilde symbol to denote variables in the dual FFG.

Here, the 1D DFT $\gamma(\cdot)$ of $v(\cdot)$ is defined as

$$\gamma(\tilde{y}) \triangleq \sum_{y \in \mathcal{X}} v(y) e^{-i2\pi \cdot y \cdot \tilde{y} / |\mathcal{X}|} \quad (27)$$

where i is the unit imaginary number [8].

We denote the partition function of the dual FFG by Z_d . According to the normal factor graph duality theorem, Z and Z_d are equal up to scale $\alpha(G) = Z_d/Z$ (see [Theorem 2][9]). The scale factor is $\alpha(G) = |\mathcal{X}|^{|E|-|V|} = |\mathcal{X}|^{|\bar{T}|-1}$, which depends on the topology of the graph. For example, in a 2D lattice with periodic boundaries $\alpha(G) = |\mathcal{X}|^{2N-N} = |\mathcal{X}|^N$.

From the primal FFG of an Ising model, we can obtain its dual by replacing each factor (8) by its 1D DFT, each equality indicator factor by a zero-sum indicator factor, and each zero-sum indicator factor by an equality indicator factor.

The dual FFGs of the Ising models in Fig. 2 are shown in Fig. 4, where the unlabeled boxes represent factors as

$$\gamma_m(\tilde{y}_m) = \begin{cases} 2 \cosh J_m, & \text{if } \tilde{y}_m = 0 \\ 2 \sinh J_m, & \text{if } \tilde{y}_m = 1 \end{cases} \quad (28)$$

which is the 1D DFT of (8), boxes labeled “+” are zero-sum indicator factors as in (9), and boxes containing “=” symbols are equality indicator factors given by (7). Notice that in a ferromagnetic model (28) is non-negative. For more details on constructing the dual FFG of the Ising model, see [10], [11], [12], [13].

By analogy with Kirchhoff’s current law, we prove:

Lemma 3. Consider a cutset in the dual FFG of the Ising model. If the variables attached to the equality indicator factors in the cutset are $\tilde{Y}_1, \tilde{Y}_2, \dots$, then

$$\sum_{m \in \text{Cutset}} \tilde{Y}_m = 0 \quad (29)$$

Proof. A cutset partitions G into $G_1 \cup G_2$. In G_1 (or in G_2), suppose we write down all the equations associated with all the zero-sum indicator factors. But the sum over all these equations in G_1 (or in G_2) is equal to zero, because each variable, say \tilde{Y}_k , appears twice in the summation. Furthermore, in G , the same sums are equal to $\sum_{m \in \text{Cutset}} \tilde{Y}_m$. ■

An example of a cutset is shown by thick edges in Fig. 5, where the variables $\tilde{Y}_1, \tilde{Y}_2, \dots$ attached to the equality indicator factors in the cutset are marked blue.

Again, let us partition G into $G = T \cup \bar{T}$, where T is a spanning tree in the dual FFG. As a result, $\tilde{\mathbf{Y}} = \tilde{\mathbf{Y}}_T \cup \tilde{\mathbf{Y}}_{\bar{T}}$. Fig. 4 shows examples of such partitionings, where cospanning trees are marked by thick blue edges, edges attached to the unlabeled boxes and to the equality indicator factors on the

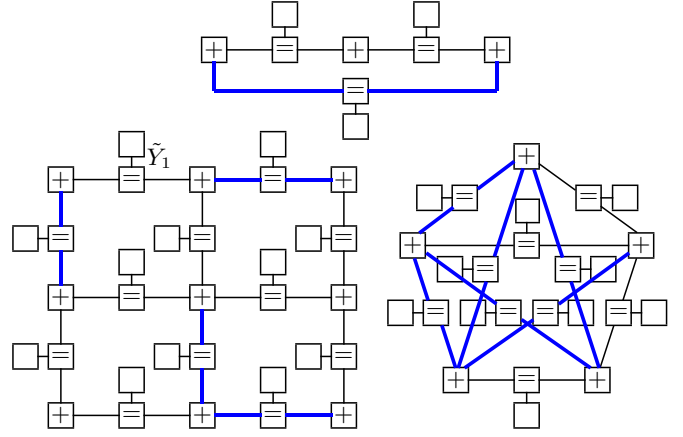


Fig. 4. Dual FFGs of the Ising models in Fig. 2. The unlabeled boxes represent (28), boxes containing “+” symbols are as in (9), and boxes containing “=” symbols are given by (7). In each dual FFG, the chords are marked by thick blue edges.

branches represent $\tilde{\mathbf{Y}}_T$, and edges attached to the unlabeled boxes and to the equality indicator factors on the chords represent $\tilde{\mathbf{Y}}_{\bar{T}}$. Although T is cycle-free, in general, \bar{T} may contain cycles (see Fig. 4–right).

Removing a branch $b \in T$ partitions $T = T_1 \cup T_2$. The edges that connect T_1 and T_2 form a unique cutset in G – called the fundamental cutset belonging to b . Each fundamental cutset has exactly one branch of T that does not appear in any other fundamental cutset². Moreover, according to Lemma 2, for each $b \in T$ we can compute \tilde{y}_b as a linear combination of $\tilde{\mathbf{Y}}_{\bar{T}}$. The size of the state space in the dual FFG is thus $|\mathcal{X}|^{|\bar{T}|}$.

Remark 2. In the dual FFG, we can freely choose a configuration $\tilde{\mathbf{y}}_{\bar{T}}$, and therefrom deterministically compute each component of $\tilde{\mathbf{y}}_T$ via its fundamental cutset.

Similar to our approach in Section II, we let

$$\Gamma_{\bar{T}}(\tilde{\mathbf{y}}_{\bar{T}}) = \prod_{m \in \bar{T}} \gamma_m(\tilde{y}_m) \quad (30)$$

$$\Gamma_T(\tilde{\mathbf{y}}_T) = \prod_{m \in T} \gamma_m(\tilde{y}_m) \quad (31)$$

and

$$\Gamma(\tilde{\mathbf{y}}) = \Gamma_T(\tilde{\mathbf{y}}_T) \cdot \Gamma_{\bar{T}}(\tilde{\mathbf{y}}_{\bar{T}}) \quad (32)$$

Since $\Gamma(\cdot)$ is non-negative, we can define the following global probability mass function in the dual FFG

$$p_d(\tilde{\mathbf{y}}) \triangleq \frac{\Gamma(\tilde{\mathbf{y}})}{Z_d} \quad (33)$$

where

$$Z_d = \sum_{\tilde{\mathbf{y}}} \Gamma(\tilde{\mathbf{y}}) \quad (34)$$

²In the dual domain, the set of all fundamental cutsets generates a vector space over \mathbb{F}_2 with dimensionality $|T|$; see [2, Chapter 2], [3, Chapter 14].

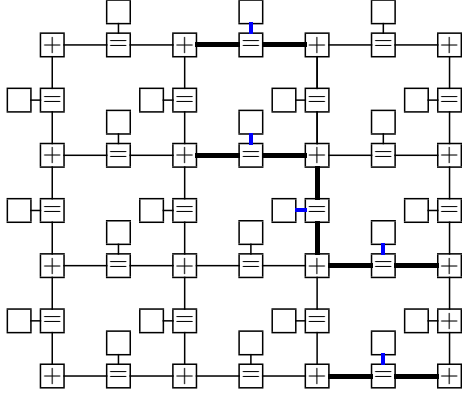


Fig. 5. The thick edges show a cutset in the dual FFG of the 2D Ising model, where variables on the cutset are marked blue.

We find it more convenient to factor out $2 \cosh J_m$ from each factor (28), and work with local factors as

$$\gamma'_m(\tilde{y}_m) = (\tanh J_m)^{\tilde{y}_m} \quad (35)$$

where the definitions of $\Gamma'_T(\cdot)$, $\Gamma'_{\bar{T}}(\cdot)$, $p'_d(\cdot)$, and Z'_d follow immediately from (35). In this set-up

$$Z_d = B \cdot Z'_d \quad (36)$$

with $B = \prod_{m \in E} (2 \cosh J_m)$.

V. IMPORTANCE SAMPLING IN THE DUAL FFG

On a cospanning tree in the dual FFG, we define the following proposal probability mass function

$$q_{\bar{T}}(\tilde{\mathbf{y}}_{\bar{T}}) \triangleq \frac{\Gamma'_{\bar{T}}(\tilde{\mathbf{y}}_{\bar{T}})}{Z_{q_{\bar{T}}}} \quad (37)$$

where $Z_{q_{\bar{T}}}$ is analytically available as

$$Z_{q_{\bar{T}}} = \sum_{\tilde{\mathbf{y}}_{\bar{T}}} \Gamma'_{\bar{T}}(\tilde{\mathbf{y}}_{\bar{T}}) = \prod_{m \in \bar{T}} (1 + \tanh J_m) \quad (38)$$

At iteration (ℓ) we first draw $\tilde{\mathbf{y}}_{\bar{T}}^{(\ell)}$ according to (37), where each component $\tilde{y}_m^{(\ell)}$ of $\tilde{\mathbf{y}}_{\bar{T}}^{(\ell)}$ is drawn independently according to $Q_m(\cdot) \propto \gamma'_m(\cdot)$. Indeed

$$Q_m(\tilde{y}_m^{(\ell)} = \zeta) = \begin{cases} \frac{1}{1 + \tanh J_m} & \text{if } \zeta = 0 \\ \frac{\tanh J_m}{1 + \tanh J_m} & \text{if } \zeta = 1 \end{cases} \quad (39)$$

We then compute $\tilde{\mathbf{y}}_T^{(\ell)}$, and finally, use L created samples in the following estimator based on importance sampling

$$\hat{Z}'_d = \frac{Z_{q_{\bar{T}}}}{L} \sum_{\ell=1}^L \Gamma'_T(\tilde{\mathbf{y}}_T^{(\ell)}) \quad (40)$$

which is unbiased, that is, $E_{q_{\bar{T}}}[\hat{Z}'_d] = Z'_d$ (see [13]).

Let us suppose that the coupling parameters on a spanning tree of the dual FFG are constant J_T . As $J_T \rightarrow \infty$, $\Gamma'_T(\cdot)$ approaches a constant factor (cf. (35)). Therefore (40) is

expected to perform well when the couplings associated with the branches of a spanning tree in the dual FFG are strong.

The variance of the estimator in (40) is given by [6]

$$\frac{L}{Z'^2_d} \text{Var}[\hat{Z}'_d] = \chi^2(p'_d, q_{\bar{T}}) \quad (41)$$

Note that $\lim_{J_T \rightarrow \infty} p'_d(\cdot) = q_{\bar{T}}(\cdot)$. This gives

$$\lim_{J_T \rightarrow \infty} \chi^2(p'_d, q_{\bar{T}}) = 0 \quad (42)$$

which suggests that a plausible strategy in partitioning the dual FFG is to include edges with stronger couplings in T . This can be done using the maximum spanning tree algorithm.

VI. CONCLUSION

We analyzed some properties of the primal and the dual FFG of the Ising model in the context of algebraic graph theory. We showed that, in the primal domain, variables can be freely chosen on a spanning tree, and the remaining variables can be computed via their fundamental cycles. Whereas in the dual domain, we can choose the variables arbitrarily on a cospanning tree, and compute the remaining variables via their fundamental cutsets. In each domain, an importance sampling algorithm was proposed to estimate the partition function. In the primal (resp. dual) FFG, the algorithm performs well when coupling parameters associated with a cospanning (resp. spanning) tree of the model are weak (resp. strong).

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